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Self-dual non-Abelian vortices in a Φ^2 Chern-Simons theory.

by

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Abstract

We study a non-Abelian Chern-Simons gauge theory in $2 + 1$ dimensions with the inclusion of an anomalous magnetic interaction. For a particular relation between the Chern-Simons (CS) mass and the anomalous magnetic coupling the equations for the gauge fields reduce from second- to first order differential equations of the pure CS type. We derive the Bogomol'nyi-type or self-dual equations for a Φ^2 scalar potential, when the scalar and topological masses are equal. The corresponding vortex solutions carry magnetic flux that is not quantized due to the non-topological nature of the solitons. However, as a consequence of the quantization of the CS term, both the electric charge and angular momentum are quantized.

keywords: Chern-Simons, non-topological, non-abelian, vortices.

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Gauge field theories present a rich spectrum of finite energy (or finite action) classical solutions; such as vortices, monopoles and instantons. These classical solutions can be classified as topological or non-topological; depending of the origin of the stability mechanism [1]. Among these theories, the self-dual theories deserve special attention. Self-duality refers to theories in which the interactions have particular forms and special strengths such that the equations of motion reduce from second- to first-order differential equations; these configurations minimize the energy (or the action). For example the Abelian-Higgs model admits topological solitons of the vortex type [2]; furthermore, when the parameters are chosen to make the vector and scalar masses equal the vortices satisfies a set of Bogomol'nyi-type or self-duality-type equations [3]. This self-dual vortex solutions have also been found for the non-Abelian Higgs theory [4]. In the self-dual point the vortices become non-interacting and static multisoliton solutions may be expected [5].

Recently there has been considerable interest in a new class of self-dual theories, the self-dual Chern-Simons (CS) theories in $(2+1)$ dimensions, which involve charged scalar fields minimally coupled to massive gauge fields whose dynamics is solely provided by the CS term, instead of the Maxwell term [6]. These theories, where the kinetic action for the gauge field is solely provided by the Chern-Simons term is known as the pure CS theory (PCS) [7, 8]. An interesting feature of these self-dual theories is that they permit a realization with either relativistic or nonrelativistic dynamics; furthermore the presence of the CS term produce interesting effects: the magnetic vortices acquire electric charge and fractional spin [9]. In the case of the relativistic theory, self-dual vortex solutions have been found for a sixth order Higgs potential in both Abelian [10, 11] and non-Abelian [12] theories.

One can pose the question whether there exist self-dual models in which the gauge field Lagrangian includes both the Maxwell and the CS term. Self-dual vortex solutions can be constructed in such a Maxwell-Chern-Simons gauge theory if one adds a magnetic moment interaction between the scalar and the gauge fields [13]. It was shown that for a special relation between the CS mass and the anomalous magnetic coupling, the equations for the gauge fields reduce from second- to first-order differential equations, similar to those of the pure CS theory. Furthermore, it

was demonstrated that non-topological charged vortices satisfy a set of Bogomol'nyi-type or self-duality equations for a quadratic potential $V(\Phi) = (m^2/2)\Phi^2$, when m and the topological masses are equal [14]. This model possess a local $U(1)$ symmetry, so we will refer to it as the Abelian Φ^2 model.

In this work we present an extension of the Φ^2 model to the non-Abelian case, finding first-order self-dual equations which can be seen to admit vortex solutions carrying magnetic flux, electric charge and spin. The same as in the Abelian model self-duality is attained for a quadratic scalar potential, consequently the solitons are non-topological; so the magnetic flux is not quantized. However, it is interesting to note that as a consequence of the quantization of the CS coefficient κ in the non-Abelian case [7], both the electric charge and the spin become quantized.

We shall consider for simplicity a non-Abelian gauge theory in $2 + 1$ dimensions with $SU(2)$ symmetry, though, the arguments that follow can be easily generalized to an $SU(N)$ theory. Thus, we have a theory of gauge fields \mathbf{A}_μ coupled to two scalar field Φ and Ψ in the adjoint representation. The Lagrangian describing our model reads

$$\begin{aligned} \mathcal{L} = & \frac{1}{2} D_\mu \Phi \cdot D^\mu \Phi + \frac{1}{2} D_\mu \Psi \cdot D^\mu \Psi - V(\Phi, \Psi) \\ & - \frac{1}{4} \mathbf{G}_{\mu\nu} \cdot \mathbf{G}^{\mu\nu} + \frac{\kappa}{4} \epsilon^{\mu\nu\alpha} \left(\mathbf{G}_{\mu\nu} \cdot \mathbf{A}_\alpha - \frac{e}{3} \mathbf{A}_\mu \cdot (\mathbf{A}_\nu \wedge \mathbf{A}_\alpha) \right), \end{aligned} \quad (1)$$

where the Minkowski-space metric is $g_{\mu\nu} = \text{diag}(+1, -1, -1)$; $\mu = (0, 1, 2)$, with

$$\mathbf{G}_{\mu\nu} = \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu + e \mathbf{A}_\mu \wedge \mathbf{A}_\nu. \quad (2)$$

In all these equations Φ , Ψ , and \mathbf{A}_μ denote triplets in isospin-space; *e.g.* $\Phi = (\phi^1, \phi^2, \phi^3)$, $\mathbf{A}_\mu = (A_\mu^1, A_\mu^2, A_\mu^3)$, etc. The covariant derivative D_μ includes both the usual minimal coupling plus the anomalous magnetic contribution:

$$D_\mu \Phi = (\partial_\mu \Phi + e \mathbf{A}_\mu \wedge \Phi + \frac{g}{4} \epsilon_{\mu\nu\alpha} \mathbf{G}^{\nu\alpha} \wedge \Phi), \quad (3)$$

with g the anomalous magnetic moment. It has been previously proved, that for Abelian gauge theories the magnetic moment interaction can be incorporated into the

covariant derivative, even for spinless particles [15, 16, 13].⁵ Here Eq. (3) shows that in the non-Abelian case the covariant coupling to the scalar field can also be modified by the inclusion of an anomalous magnetic term. This extra term is consistent with the Lorentz and gauge covariance of the theory, but give rise to the breaking of P and T symmetries.

We choose for the potential

$$V(\Phi, \Psi) = V_1(\Phi) + V_2(\Psi) + \lambda (\Phi \cdot \Psi)^2, \quad (4)$$

where the last term is selected in such a way that the energy is minimized by configurations in which Φ and Ψ are orthogonal in isospin space. We also require $V_2(\Psi)$ to be minimized at a non-zero value $\Psi^2 = v^2$, *i.e.* $V_2(|\Psi| = v) = 0$. For the moment we leave $V_1(\Phi)$ free; we shall see that it is determined by the Bogomol'nyi equation to be $V_1(\Phi) = (\kappa/2) \Phi^2$.

The field Lagrangian in Eq. (1) leads to gauge covariant equations of motion. However, the Lagrangian itself is not gauge invariant. Indeed, as a response of a gauge transformation U , the change in the action S is $S \rightarrow S + \kappa(8\pi^2/e^2)\omega(U)$, where $\omega(U)$ is the winding number of the gauge transformation. The corresponding quantum theory is well defined if the change in the action is a multiple of 2π ; this leads to the quantization condition of the topological mass [7]:

$$\kappa = \frac{e^2}{4\pi} n, \quad n \in \mathbf{Z}. \quad (5)$$

The equations of motion for the Lagrangian in Eq. (1) are

$$D_\mu D^\mu \Phi = -\frac{\delta V}{\delta \Phi}, \quad D_\mu D^\mu \Psi = -\frac{\delta V}{\delta \Psi}, \quad (6)$$

$$\epsilon_{\mu\nu\alpha} \nabla^\mu [\mathbf{G}^\alpha + \frac{g}{2e} \mathbf{J}^\alpha] = \mathbf{J}_\nu - \kappa \mathbf{G}_\nu. \quad (7)$$

The last equation has been written in terms of the dual field, $\mathbf{G}_\mu \equiv \frac{1}{2} \epsilon_{\mu\nu\alpha} \mathbf{G}^{\nu\alpha}$. We notice that in the previous equation ∇^μ includes only the contribution on the gauge

⁵The magnetic moment interaction term in CS gauge theories has also been considered in some other context, see references [17].

potential (compare with the full covariant derivative Eq. (3)):

$$\nabla_\mu \Phi = (\partial_\mu \Phi + e \mathbf{A}_\mu \wedge \Phi) , \quad (8)$$

and the conserved matter current is given by

$$\mathbf{J}_\mu = e (D_\mu \Phi \wedge \Phi + D_\mu \Psi \wedge \Psi) . \quad (9)$$

In the case of the Abelian theory the gauge field equations reduce from second- to first-order differential equations [15, 13], similar to those of the PCS type [8] when the relation $\kappa = -\frac{2e}{g}$ holds. The same is true in the non-Abelian theory; indeed we notice that if the relation

$$\kappa = -\frac{2e}{g} \quad (10)$$

holds, then it is clear that the Eq. (7) is solved identically if we choose the first order ansatz

$$\mathbf{G}_\mu = \frac{1}{\kappa} \mathbf{J}_\mu , \quad (11)$$

that has the same structure as the equations of the PCS theory [8]. We will refer to the previous conditions as the PCS limit. However, we should notice that the explicit expression for \mathbf{J}_μ differs from the usual expression of the PCS theory, because according to Eqs. (9) and (3) \mathbf{J}_μ receives contributions from the anomalous magnetic moment. This PCS equations (Eq. (11)) imply that any object carrying magnetic flux (Φ_B) must also carry electric charge (Q), with the two quantities related as

$$Q = \kappa \Phi_B . \quad (12)$$

In what follows we shall work in the limit in which Eqs. (11) and (10) are valid, so we consider Eq. (11) as the equation of motion for the gauge fields, instead of Eq. (7).

In the so called Bogolmol'nyi limit all the equations of motion are known to become first order differential equations [3]; furthermore, it is possible to write the equations of motion as self-duality equations. The existence of this limit usually requires a

specific form for the scalar potential. To look for the Bogomol'nyi equations we start from the energy-momentum tensor that is obtained by varying the curved-space form of the action with respect to the metric; this yields

$$\begin{aligned}
T_{\mu\nu} = & \left[\delta_{ab} - \frac{g^2\Phi^2}{4} (\delta_{ab} - \hat{\phi}_a\hat{\phi}_b) - \frac{g^2\Psi^2}{4} (\delta_{ab} - \hat{\psi}_a\hat{\psi}_b) \right] \left(G_\mu^a G_\nu^b - \frac{1}{2} g_{\mu\nu} G_\alpha^a G_b^\alpha \right) \\
& + \nabla_\mu \Phi \cdot \nabla_\nu \Phi - \frac{1}{2} g_{\mu\nu} (\nabla_\lambda \Phi)^2 + \nabla_\mu \Psi \cdot \nabla_\nu \Psi - \frac{1}{2} g_{\mu\nu} (\nabla_\lambda \Psi)^2 + g_{\mu\nu} V(\Phi, \Psi),
\end{aligned} \tag{13}$$

where $\hat{\phi}_a = \phi_a/|\Phi|$, $\hat{\psi}_a = \psi_a/|\Psi|$ and we recall that ∇_μ includes only the gauge potential contribution (Eq. (8)); we reserve the letters a, b, c, \dots for isospin indices and i, j, k, \dots for space indices. Notice that both the Chern-Simons and linear terms in g do not appear explicitly in $T_{\mu\nu}$. This is a consequence of the fact that these terms do not make use of the space-time metric tensor $g_{\mu\nu}$; then, when $g_{\mu\nu}$ is varied to produce $T_{\mu\nu}$ no contributions arise from these terms [8].

As it is well-known, finite energy determines the asymptotic behavior of the fields at spatial infinity. Thus, we demand that every term in Eq. (13) vanish as $\rho \rightarrow \infty$:

$$\begin{aligned}
\lim_{\rho \rightarrow \infty} \nabla_\mu \Phi &= 0, & \lim_{\rho \rightarrow \infty} \nabla_\mu \Psi &= 0, \\
\lim_{\rho \rightarrow \infty} \mathbf{G}_\mu &= 0, & \lim_{\rho \rightarrow \infty} V(\Phi, \Psi) &= 0.
\end{aligned} \tag{14}$$

The equations of motion and the energy-momentum tensor are too complicated in their actual form. So, in order to find static finite-energy solutions we shall assume some simplifying properties for the fields; later, an ansatz consistent with these conditions will be presented. We look for conditions in such a way that the field Ψ plays no dynamical role, but it points along a constant direction in isospin-space that can be used to define a gauge invariant Abelian field strength. Thus, we take the field Ψ as a constant everywhere in space; we expect that any other configuration with a non-constant Ψ should lead to a greater energy. Furthermore, we also assume that the field Ψ is parallel in isospin space to the gauge fields. We shall then look for the self-duality equations under the following conditions (valid everywhere):

$$\begin{aligned}
D_\mu \Psi &= 0, & \Psi^2 &= v^2, \\
\Psi \cdot \Phi &= 0, & \Psi &\parallel \mathbf{A}_\mu.
\end{aligned} \tag{15}$$

Notice that under these conditions the last two terms of the potential (Eq. (4)) vanish identically and that the field Ψ does not contribute to the total energy. Using the previous conditions and Eqs. (9) and (11) one can also see that $\Phi \cdot \mathbf{G}_\mu = 0$. Due to the fact that Φ and \mathbf{G}_μ are mutually orthogonal we can not project \mathbf{G}_μ along the direction of Φ in order to define the physically observable electromagnetic field $F_{\mu\nu}$. This is the reason why we introduced a second scalar field Ψ ; hence we propose to construct the gauge invariant Abelian field strength by setting

$$F_{\mu\nu} = \hat{\Psi} \cdot \mathbf{G}_{\mu\nu}, \quad \hat{\Psi} = \frac{\Psi}{|\Psi|}. \tag{16}$$

The energy density (T_{00}) for a static configuration can be written in terms of the “cromo-electric” $E_i^a = -\epsilon_{ij} G_j^a$ and “cromo-magnetic” $B^a = G_0^a$ fields as

$$T_{00} = \frac{1}{2} \left(1 - \frac{e^2}{\kappa^2} \Phi^2 \right) (\mathbf{E}_i^2 + \mathbf{B}^2) + \frac{1}{2} (\mathbf{A}_0 \wedge \Phi)^2 + \frac{1}{2} (\nabla_i \Phi)^2 + V_1(\Phi). \tag{17}$$

Defining a reduced gauge potential

$$\bar{\mathbf{A}}_\mu = \mathbf{A}_\mu + \frac{1}{e} \hat{\Phi} \wedge \partial_\mu \hat{\Phi}, \tag{18}$$

with $\hat{\Phi} = \Phi/|\Phi|$, we can combine Eqs. (11) and (9) to express \mathbf{G}_μ in terms of $\bar{\mathbf{A}}_\mu$

$$G_\mu^a = -\kappa \Phi^2 C \left(\delta_{ab} - \hat{\phi}_a \hat{\phi}_b \right) \bar{A}_\mu^b, \tag{19}$$

where $C = e^2/(\kappa^2 - e^2 \Phi^2)$. Notice, that Eq. (19) is in agreement with the condition $\Phi \cdot \mathbf{G}_\mu = 0$, therefore $\Phi \cdot \mathbf{B} = 0$ and $\Phi \cdot \mathbf{E}_i = 0$.

We can now use the $\mu = 0$ component of the previous equation to eliminate \mathbf{A}_0 from Eq. (17). Hence, the part of T_{00} depending on \mathbf{B} and \mathbf{A}_0 can be recast as

$$\frac{1}{2} \left(1 - \frac{e^2}{\kappa^2} \Phi^2 \right) \mathbf{B}^2 + \frac{1}{2} (\mathbf{A}_0 \wedge \Phi)^2 = \frac{1}{2C\Phi^2} \mathbf{B}^2, \tag{20}$$

where we have used the condition $\Phi \cdot \mathbf{B} = 0$.

Likewise, using the $\mu = i$ components of Eq. (19) we can combine the parts of T_{00} involving the electric field and $\nabla_i \Phi$ into the following form

$$\frac{1}{2} \left(1 - \frac{e^2}{\kappa^2} \Phi^2 \right) (E_i^a)^2 + \frac{1}{2} (\nabla_i \Phi)^2 = \frac{1}{2\Phi^2} (\Phi \cdot \partial_i \Phi)^2 + \frac{1}{2} \kappa^2 \Phi^2 C (\delta_{ab} - \hat{\phi}_a \hat{\phi}_b) \bar{A}_i^a \bar{A}_i^b. \quad (21)$$

Substituting the results of Eqs. (20) and (21) into Eq. (17) we can write down the energy $E = \int d^2x T_{00}$ as

$$E = \int d^2x \left(\frac{1}{2C\Phi^2} \mathbf{B}^2 + \frac{1}{2} \left[\frac{1}{\Phi^2} (\Phi \cdot \partial_i \Phi)^2 + \kappa^2 \Phi^2 C (\delta_{ab} - \hat{\phi}_a \hat{\phi}_b) \bar{A}_i^a \bar{A}_i^b \right] + V_1(\Phi) \right). \quad (22)$$

The energy written in this form is similar to the expression that appears in the Nielsen-Olesen model. Thus, starting from Eq. (22) we can follow the usual Bogomol'nyi-type arguments in order to obtain the self-dual limit. The energy may then be rewritten, after an integration by parts, as

$$\begin{aligned} E = & \frac{1}{2} \int d^2x \left[\frac{1}{C\Phi^2} (B_a \mp \kappa \sqrt{C} \Phi^2 \hat{\psi}_a)^2 + |\psi_a \frac{\partial_{\pm} \Phi^2}{2|\Phi|} - i\kappa |\Phi| \sqrt{C} (\delta_{ab} - \hat{\phi}_a \hat{\phi}_b) \bar{A}_{\pm}^b|^2 \right] \\ & + \int d^2x \left[V_1(\Phi) - \frac{1}{2} \kappa^2 \Phi^2 \right] \pm \int d^2x \psi_a \left[\frac{\kappa}{\sqrt{C}} B_a - \frac{\kappa \sqrt{C}}{2} \epsilon_{ij} (\partial_i \Phi^2) \bar{A}_j^a \right], \quad (23) \end{aligned}$$

where $\partial_{\pm} = \partial_1 \pm i\partial_2$, $\bar{\mathbf{A}}_{\pm} = \bar{\mathbf{A}}_1 \pm i\bar{\mathbf{A}}_2$ and conditions (15) have been repeatedly used. Following [4] we choose a gauge in such a way that the conditions

$$\epsilon_{ij} \hat{\Psi} \cdot (\mathbf{A}_i \wedge \mathbf{A}_j) = 0, \quad \epsilon_{ij} \hat{\Psi} \cdot (\partial_i \hat{\Phi} \wedge \partial_j \hat{\Phi}) = 0 \quad (24)$$

hold. Hence, recalling the definition of the Abelian field strength Eq. (16), one has that the last two terms in Eq. (23) can be related to the magnetic flux of the vortex configuration

$$\begin{aligned}
& \pm \int d^2x \psi_a \left[\frac{\kappa}{\sqrt{C}} B_a - \frac{\kappa\sqrt{C}}{2} \epsilon_{ij} (\partial_i \Phi^2) \bar{A}_j^a \right] = \pm \kappa \int d^2x \psi^a \epsilon_{ij} \partial_i \left[\frac{1}{\sqrt{C}} A_j^a + \Lambda_j^a \right] = \\
& \pm \kappa \oint_{r=\infty} \Psi \cdot \left(\frac{1}{\sqrt{C}} \mathbf{A}_i + \mathbf{\Lambda}_i \right) dl_i = \pm \frac{\kappa^2}{e} \oint_{r=\infty} \Psi \cdot \mathbf{A}_i dl_i = \pm \frac{\kappa^2}{e} \int d^2x \Psi \cdot \mathbf{B} \equiv \frac{\kappa^2}{e} |\Phi_B|,
\end{aligned} \tag{25}$$

where we defined $\Lambda_i^a = -e^2 \left[\kappa - \sqrt{\kappa^2 - e^2 \Phi^2} \right] \epsilon_{abc} \hat{\phi}_b \partial_i \hat{\phi}_c$ and we have used the gauge conditions (24) and the fact that for any non-topological soliton the asymptotic conditions are such that $\Phi \rightarrow 0$ at spatial infinity. Thus, along the line integral: Λ_i^a vanishes and $1/\sqrt{C} = \sqrt{\kappa^2 - e^2 \Phi^2}/e \rightarrow \kappa/e$.

We then see from Eqs. (23) and (25) that the energy is bounded below; for a fixed value of the magnetic flux, the lower bound is given by $E \geq \frac{\kappa^2}{e} \Phi_B$ provided that the potential $V_1(\Phi)$ is chosen as $V_1(\Phi) = \frac{m^2}{2} \Phi^2$ with the critical value $m = \kappa$, *i.e.* when the scalar and the topological masses are equal. Therefore, in this limit we are necessarily in the symmetric phase of the theory. From Eq. (23) we see that the lower bound for the energy

$$E = \frac{\kappa^2}{e} |\Phi_B| = \frac{\kappa}{e} |Q|, \tag{26}$$

is saturated when the following self-duality equations are satisfied:

$$B_a = \pm \frac{\kappa e \Phi^2}{[\kappa^2 - e^2 \Phi^2]^{1/2}} \hat{\psi}_a, \tag{27}$$

$$\frac{1}{2} \partial_{\pm} \Phi^2 = \frac{ie\kappa \Phi^2}{[\kappa^2 - e^2 \Phi^2]^{1/2}} \hat{\psi}_a \bar{A}_{\pm}^a, \tag{28}$$

where the upper (lower) sign corresponds to positive (negative) value of the magnetic flux. Eq. (27) implies that the magnetic field vanishes whenever Φ does. The finiteness energy condition forces the scalar field to vanish both at the center of the vortex and also at spatial infinity; consequently as it happens in the Abelian case [13], the magnetic flux of the vortices lies in a ring. It is interesting to remark that Eq. (28) can be written as a self-duality equation; indeed if we define a new covariant derivative as $\tilde{D}_i = \partial_i - i2e\kappa/\sqrt{\kappa^2 - e^2 \Phi^2}$, then Eq. (28) is equivalent to

$$\tilde{D}_i \Phi^2 = \mp i \epsilon_{ij} \tilde{D}_j \Phi^2. \quad (29)$$

Eqs. (27) and (28) can be reduced to one nonlinear second order differential for one unknown function. To do this, first notice that Eq. (28) implies that $\hat{\Psi} \cdot \bar{\mathbf{A}}_i$ can be determined in terms of the scalar field as

$$\hat{\Psi} \cdot \bar{\mathbf{A}}_i = \pm \frac{1}{2\kappa\sqrt{C}} \epsilon_{ij} \partial_j \ln(\Phi^2); \quad (30)$$

when this equation is substituted into Eq. (27), we get

$$\partial_i \left[\frac{\sqrt{\kappa^2 - e^2 \Phi^2}}{2e\kappa} \partial_i \ln(\Phi^2) \right] \mp \frac{1}{e} \epsilon_{ij} \hat{\Psi} \cdot (\partial_i \hat{\Phi} \wedge \partial_j \hat{\Phi}) - \frac{\kappa e \Phi^2}{\sqrt{\kappa^2 - e^2 \Phi^2}} = 0. \quad (31)$$

This equation still involves the three components of the scalar field (Φ), but taking into account the gauge condition $\epsilon_{ij} \hat{\Psi} \cdot (\partial_i \hat{\Phi} \wedge \partial_j \hat{\Phi}) = 0$ and defining $\sigma = e^2 \Phi^2$ we find that Eq. (31) is a second order nonlinear differential equation for σ :

$$\partial_i \left[\sqrt{\kappa^2 - \sigma} \partial_i \ln \sigma \right] - \frac{2\kappa^2 \sigma}{\sqrt{\kappa^2 - \sigma}} = 0. \quad (32)$$

If we assume a rotationally invariant form for Φ^2 , this equation reduces to the Equation (16) of reference [13] and therefore we expect that the same features of the Abelian vortex solution are present in the non-Abelian case. However, we first have to show that it is possible to find a vortex configuration that is consistent with the conditions imposed on Ψ (15) and the gauge fixing conditions (24). In this respect the ansatz of reference [12] fulfils these requirements; this axially symmetric ansatz is given by

$$\begin{aligned} \Phi &= \frac{\kappa}{e} f(\rho) (\cos \theta, \sin \theta, 0), & \Psi &= v \hat{e}_3, \\ \mathbf{A}_\theta &= \hat{e}_3 \frac{a(\rho) - 1}{e\rho}, & \mathbf{A}_0 &= \hat{e}_3 \frac{\kappa}{e} h(\rho), \end{aligned} \quad (33)$$

where (ρ, θ) are polar coordinates and $\hat{e}_3 = (0, 0, 1)$ is a unit vector in isospin-space. It is straightforward to check that this ansatz satisfies the conditions (15) and (24).

Substitution of this ansatz into Eqs. (27) and (28) leads to radial equations that coincides with those found for the Abelian case [13]

$$\frac{da}{d\rho} = \pm \frac{\kappa^2 \rho f^2}{(1-f^2)^{1/2}}, \quad \frac{df}{d\rho} = \mp \frac{fa}{\rho(1-f^2)^{1/2}}. \quad (34)$$

All the features of the non-topological vortex solution thus corresponds, for this special ansatz, to those found in [13] if the vorticity number n is chosen as $n = 1$. In particular, the magnetic flux for these solutions is concentrated in a ring surrounding the center of the vortex. The solution is characterized by a real valued constant α , that is related with the asymptotic behavior of the fields. Near the origin the fields behave as: $f \rightarrow \rho$, $a \rightarrow 1$; whereas at spatial infinity we have $f \rightarrow \rho^{-\alpha}$, $a \rightarrow -\alpha$. The parameter α satisfies the bounds $1 < \alpha < 3$ [18]. Once that the boundary conditions are known the magnetic flux can be calculated, using the ansatz Eq. (33) and the definition of the Abelian field strength Eq. (16), as $\Phi_B = (2\pi/e)(1 + \alpha)$; notice that the magnetic flux is not quantized. The soliton is also characterized by a charge, that can be directly computed from the fundamental relation Eq. (12), it results in

$$Q = \frac{2\pi\kappa}{e}(1 + \alpha) = \frac{e}{2}(1 + \alpha)n, \quad (35)$$

where Eq. (5) was used to write the last equality and we recall that n is an integer associated with the quantization of the CS term. We can also compute the angular momentum (spin) of the vortex. In $(2 + 1)$ dimensions there is only one generator of angular momentum $J = \int d^2x \epsilon^{ij} x_i T_{0j}$; using the expression (13) for the energy-momentum tensor, J becomes

$$J = \frac{\pi\kappa}{e^2}(\alpha^2 - 1) = \frac{1}{4}(\alpha^2 - 1)n. \quad (36)$$

Let us notice that due to the non-topological nature of the solitons the magnetic flux is not quantized. However, in the present non-Abelian model both the electric charge Q and the angular momentum J are quantized, a condition that follows from the quantization of the CS coefficient Eq. (5). Here, we conclude with some comments in relation to the statistical parameter $\Delta\theta$. According to the fundamental CS relation Eq. (12) in $2 + 1$ dimensions, each charge becomes a flux tube, and viceversa. Thus,

the effect of the CS term is to transmute the statistics of the particles. Indeed the statistical parameter $\Delta\theta$, arising when objects carrying magnetic flux and electric charge winds around another, is given by [19]

$$\Delta\theta = \frac{1}{2}Q\Phi_B = \frac{1}{2}\frac{Q^2}{\kappa} = \frac{1}{2}\kappa\Phi_B^2. \quad (37)$$

This relation leads to fractional statistics. For the present non-topological solitons and using Eqs. (5) and (35) the statistical parameter results in $\Delta\theta = (n\pi/2)(1 + \alpha)^2$. Then, according to Eq. (36) the statistical parameter is related to the spin as

$$\Delta\theta = 2\pi\frac{(\alpha + 1)}{(\alpha - 1)}J, \quad (38)$$

instead of the usual spin-statistics relation $|\Delta\theta| = 2\pi J$. We recall that α is related to the asymptotic value of the gauge field; for topological solitons $a(\rho)$ in Eq. (33) vanishes at spatial infinity, so the spin-statistics relation holds [12]. Instead, for non-topological solitons the value $a(\infty) = -\alpha$ is not fixed and leads to the modification of the spin-statistics relation according to Eq. (38).

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